Linear Algebra Done Right

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Fourth Edition

Contents

1 Vector Spaces

1A *Exercises*

Problem 1A.1. *Show that* $\alpha + \beta = \beta + \alpha$ *for all* α, β *in* $\mathbb{C}$

Proof. Let $\alpha = a + bi$ and $\beta = c + di$ for $a, b, c, d \in \mathbb{R}$. We have

$$\begin{aligned} \alpha + \beta &= (a + bi) + (c + di) \\ &= (a + c) + (b + d)i \\ &= (c + a) + (d + b)i \\ &= (c + di) + (a + bi) \\ &= \beta + \alpha \end{aligned}$$

□

Problem 1A.2. *Show that* $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ *for all* $\alpha, \beta, \gamma \in \mathbb{C}$.

Proof. Let $\alpha = a_1 + a_2 i, \beta = b_1 + b_2 i$, and $\gamma = c_1 + c_2 i$ for all $a_k, b_k, c_k \in \mathbb{R}$ where $k \in \{1, 2\}$. We have

$$\begin{aligned} (\alpha + \beta) + \gamma &= \big[(a_1 + a_2 i) + (b_1 + b_2 i)\big] + (c_1 + c_2 i) \\ &= \big[(a_1 + b_1) + (a_2 + b_2)i\big] + (c_1 + c_2 i) \\ &= \big[(a_1 + b_1) + c_1\big] + \big[(a_2 + b_2) + c_2\big]i \\ &= \big[a_1 + (b_1 + c_1)\big] + \big[a_2 + (b_2 + c_2)\big]i \\ &= (a_1 + a_2 i) + \big[(b_1 + c_1) + (b_2 + c_2)i\big] \\ &= (a_1 + a_2 i) + \big[(b_1 + b_2 i) + (c_1 + c_2 i)\big] \\ &= \alpha + (\beta + \gamma) \end{aligned}$$

□

Problem 1A.3. *Show that* $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ *for all* $\alpha, \beta, \gamma \in \mathbb{C}$.

Proof. Let $\alpha = a_1 + a_2 i, \beta = b_1 + b_2 i$, and $\gamma = c_1 + c_2 i$ for all $a_k, b_k, c_k \in \mathbb{R}$ where $k \in \{1, 2\}$. We have

$$\begin{aligned}
(\alpha\beta)\gamma &= \big[(a_1 + a_2 i)(b_1 + b_2 i)\big](c_1 + c_2 i) \\
&= \big[(a_1 b_1 - a_2 b_2) + (a_1 b_2 + a_2 b_1) i\big](c_1 + c_2 i) \\
&= \big[(a_1 b_1 - a_2 b_2) c_1 - (a_1 b_2 + a_2 b_1) c_2\big] + \big[(a_1 b_1 - a_2 b_2) c_2 + (a_1 b_2 + a_2 b_1) c_1\big] i \\
&= (a_1 b_1 c_1 - a_2 b_2 c_1 - a_1 b_2 c_2 - a_2 b_1 c_2) + (a_1 b_1 c_2 - a_2 b_2 c_2 + a_1 b_2 c_1 + a_2 b_1 c_1) i \\
&= \big[(a_1 b_1 c_1 - a_1 b_2 c_2) - (a_2 b_1 c_2 + a_2 b_2 c_1)\big] + \big[(a_1 b_1 c_2 + a_1 b_2 c_1) + (a_2 b_1 c_1 - a_2 b_2 c_2)\big] i \\
&= \big[a_1 (b_1 c_1 - b_2 c_2) - a_2 (b_1 c_2 + b_2 c_1)\big] + \big[a_1 (b_1 c_2 + b_2 c_1) + a_2 (b_1 c_1 - b_2 c_2)\big] i \\
&= (a_1 + a_2 i)\big[(b_1 c_1 - b_2 c_2) + (b_2 c_1 + b_1 c_2) i\big] \\
&= (a_1 + a_2 i)\big[(b_1 + b_2 i)(c_1 + c_2 i)\big] \\
&= \alpha(\beta\gamma)
\end{aligned}$$

□

Problem 1A.4. *Show that* $\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta$ *for all* $\gamma, \alpha, \beta \in \mathbb{C}$.

Proof. Let $\alpha = a_1 + a_2 i, \beta = b_1 + b_2 i$, and $\gamma = c_1 + c_2 i$ for all $a_k, b_k, c_k \in \mathbb{R}$ where $k \in \{1, 2\}$. We have

$$\gamma(\alpha + \beta) = (c_1 + c_2 i)\big[(a_1 + a_2 i) + (b_1 + b_2 i)\big]$$

□

Problem 1A.5. *Show that for every* $\alpha \in \mathbb{C}$*, there exists a unique* $\beta \in \mathbb{C}$ *such that* $\alpha + \beta = 0$.

Proof.

□

Problem 1A.6. *Show that for every* $\alpha \in \mathbb{C}$ *with* $\alpha \neq 0$, *there exists a unique* $\beta \in \mathbb{C}$ *such that* $\alpha\beta = 1$.

Proof.

□

Problem 1A.7. *Show that*

$$\frac{-1+\sqrt{3}i}{2}$$

is a cube root of 1 (meaning that its cube equals 1).

Proof.

□

Problem 1A.8. *Find two distinct square roots of* i.

Proof.

□

Problem 1A.9. *Find* $x \in \mathbb{R}^4$ *such that*

$$(4, -3, 1, 7) + 2x = (5, 9, -6, 8)$$

Proof. □

Problem 1A.10. *Explain why there does not exist $\lambda \in \mathbb{C}$ such that*

$$\lambda(2, -3i, 5+4i, -6+7i) = (12-5i, 7+22i, -32-9i)$$

Proof. □

1B *Exercises*

1C *Exercises*

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